# Math 2002 Midterm: Solutions

1. (a)

$$\int_D f(x,y)dA = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

   (b) If $C$ is a smooth simple closed curve, positively oriented, $D$ the region bounded by $C$, and $P$ and $Q$ have continuous partial derivatives, then

$$\int_C P\,dx + Q\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

2. The region is contained in $[0,3]\times[0,3]$, so let us use that as our bounding rectangle. If we use $n = 3, m = 3$, and lower-left endpoints, then only the points

$$(0,0), (1,0), (2,0), (1,1), (1,2), (2,2)$$

   are in the region $D$. Of these points, only the last three give non-zero values when put into the function. Finally, note that $\Delta A = 1$. Thus our estimate is

$$f(1,1) + f(1,2) + f(2,2) = e + e^4 + 2e^4 = e + 3e^4$$

3. In the yz-plane, the region is bounded by $y^2 + z^2 = 4$, and the bounds for $x$ are given by $0 \le x \le 3 - z$. The volume is thus

$$\int_D \int_0^{3-z} 1\,dx\,dA = \int_D 3 - z\,dA$$

   To evaluate the double integral, we switch to polar co-ordinates. Then the above becomes

$$\begin{aligned}
&= \int_0^{2\pi}\int_0^2 (3 - r\cos\theta)r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2 3r - r^2\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{3r^2}{2} - \frac{r^3\cos\theta}{3}\right)_0^2 d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{2\pi} 6 - \frac{8\cos\theta}{3}\, d\theta \\
&= \left(6\theta - \frac{8\sin\theta}{3}\right)_0^{2\pi} \\
&= 12\pi
\end{aligned}
$$

4. (a) Since

$$\frac{\partial Q}{\partial x} = -\sin(xy) - xy\cos(xy) \text{ and } \frac{\partial P}{\partial y} = -\sin(xy) - xy\cos(xy)$$

this vector field is conservative. If $\bigtriangledown f = \mathbf{F}$, then

$$\frac{\partial f}{\partial x} = 2e^{2x} - y\sin(xy) \text{ and } \frac{\partial f}{\partial y} = -x\sin(xy) + e^{2y}$$

Integrating the first with respect to $x$ gives

$$f = e^{2x} + \cos(xy) + g(y)$$

Differentiating this with respect to $y$ gives

$$\frac{\partial f}{\partial y} = -x\sin(xy) + g'(y)$$

Comparing this with our expression above gives $g'(y) = e^{2y}$, so $g(y) = \frac{e^{2y}}{2} + C$. Thus

$$f = e^{2x} + \cos(xy) + \frac{e^{2y}}{2}$$

is a potential function.

(b) Since

$$\frac{\partial Q}{\partial x} = \frac{e^{xy}}{y} \text{ and } \frac{\partial P}{\partial y} = \frac{e^{x}}{y}$$

the vector field is not conservative.

5. The curve has parametrization $x = 3\cos\theta, y = 3\sin\theta$, with $0 \leq \theta \leq \pi$. Thus the line integral becomes

$$\int_0^{\pi} (3\cos\theta)^2 (3\sin\theta)\sqrt{(-3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$

$$
\begin{aligned}
&= \int_0^{\pi} 3^3 \cos^2\theta \sin\theta \sqrt{9(\sin^2\theta + \cos^2\theta)}\ d\theta \\
&= \int_0^{\pi} 3^4 \sin\theta \cos^2\theta\ d\theta \\
&= 3^4 \left(\frac{-\cos^3\theta}{3}\right)_0^{\pi} \\
&= 27(-(-1) - (-1)) \\
&= 54
\end{aligned}
$$

6. Since the curve is closed, we can use Green's theorem. Note that the curve is negatively oriented, so the integral becomes

$$
-\int_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\, dA = -\int_D 4y - 2y\, dA = -\int_D 2y\, dA
$$

The region $D$ is bounded by the curves $y = 2x$, $x + y = 3$, and $y = 0$. If one graphs the region, one can see that it is type 2, with $0 \leq y \leq 2$ and $\frac{y}{2} \leq x \leq 3 - y$. So the integral becomes

$$
\begin{aligned}
&- \int_0^2 \int_{\frac{y}{2}}^{3-y} 2y\, dx\, dy \\
&= - \int_0^2 2y \left(3 - y - \frac{y}{2}\right) dx\, dy \\
&= - \int_0^2 6y - 3y^2\, dy \\
&= - \left(3y^2 - y^3\right)_0^2 \\
&= - (12 - 8) \\
&= -4
\end{aligned}
$$

7. To evaluate the work done, we must find the value of the line integral $\int_C \mathbf{F} \cdot dr$. Since $C$ is not closed and $\mathbf{F}$ is not conservative, we use the basic form for evaluating line integrals of vector fields.

$$
\begin{aligned}
&\int_C P\, dx + Q\, dy \\
&= \int_0^1 -(t^2 + 4)(3t^2) + (t^3)(2t)\, dt \\
&= \int_0^1 -3t^4 - 12t^2 + 2t^4\, dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^1 -t^4 - 12t^2\,dt \\
&= \left(\frac{-t^5}{5} - 4t^3\right)_0^1 \\
&= \frac{-21}{5}
\end{aligned}
$$

Since the value is negative, the wind is hindering the ant's progress.

One can also check that the wind is hindering the ant by graphing both the ant's path and the vector field, and observe that the wind is blowing against the ant's path.

8. (Bonus question) The fundamental theorem for line integrals says that for $f$ a function of two or three variables,

$$\int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

We prove this by starting with the left side:

$$
\begin{aligned}
& \int_C \nabla f \cdot dr \\
=\;& \int_C P\,dx + Q\,dy + R\,dz \\
=\;& \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\right) dt \\
=\;& \int_a^b f'(r(t))\,dt \text{ (by the chain rule)} \\
=\;& f(r(b)) - f(r(a)) \text{ (by the fundamental theorem of calculus)}
\end{aligned}
$$

Thus we have proven the required result.